Pick's Theorem: Drawing Fun Pics with PickDraw

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1 Introduction

Almost everyone has had this experience—sitting with a piece of graph paper and a pencil, with nothing else to do. With boundless childhood imagination, many of us have connected the intersections of the grid lines to form intriguing shapes and figures. Perhaps you may have even wondered what the area of such a figure was.

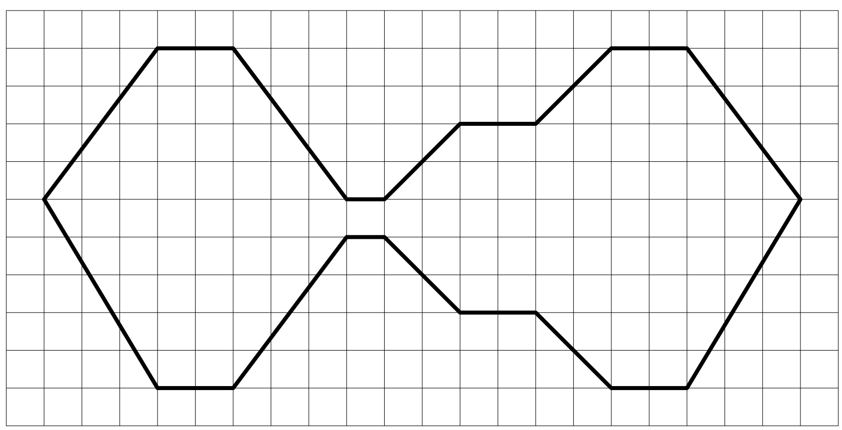


Figure 1. A lattice point polygon representing a *Star Wars* TIE fighter.

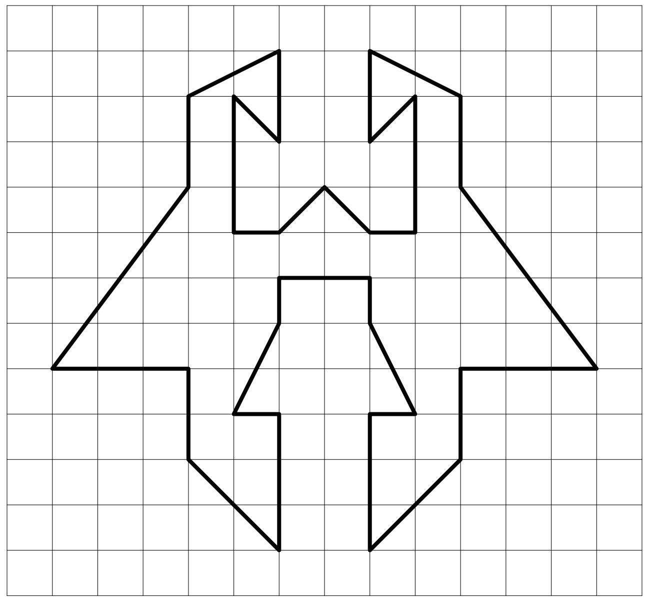


Figure 2. A simplified polygonal *Darth Vader* mask.

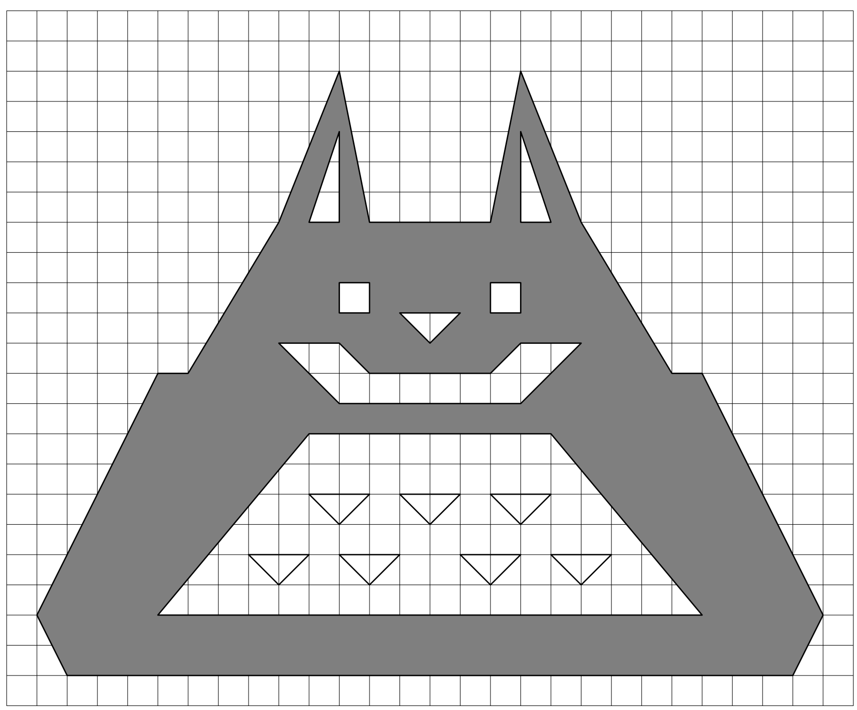


Figure 3. A lattice polygon rendition of Totoro, a Japanese cartoon character.

There are countless ways to calculate the areas of figures like these: one could use the Shoelace Theorem, for example, but this would require finding the coordinates of all of the points used and then performing tedious calculations on them. One could also attempt to use triangulation to calculate the area of the figure: this is theoretically possible, but practically infeasible. So, is there a simple way to calculate areas of lattice polygons?

This is the fundamental question resolved by Pick's theorem, which gives a simple formula to calculate the area of a lattice polygon. The formula requires only the number of boundary points and interior points, which are relatively easy to count. The condition of the pictures is simply that it is a closed polygon.

Pick's Theorem is not only surprisingly elegant in how simple the formula is, but it also has a wide range of applications such as in Farey sequences, Ford circles, and more. Pick's theorem can also be used to prove several other interesting proofs, too, such as the curious fact that the area of a lattice pentagon must always be at least 5.5.

2 Pick's Theorem

Pick's Theorem was introduced by George Alexander Pick, an Austrian mathematician who spent most of his career in Prague. It was first published in 1899 although, similar to many other great theorems, it did not receive much attention. After it was included in Steinhaus' 1969 Mathematical Snapshots, the theorem started attracting admiration for its simplicity, elegance, and power.

So what exactly does Pick's Theorem state? What is the area of such a drawing (or lattice polygon in general)? It simply states that the area of a polygon with vertices at lattice points in a lattice plane is equal to one less than the number of interior points added to half of the number of boundary points.

Pick's Theorem. Given a polygon whose vertices lie solely on lattice points, the area A of such a polygon is equal to

$$I + \frac{1}{2}B - 1,$$

where I is the number of lattice points that lie completely within the polygon and B is the number of points that lie on the edges of the polygon.

For example, the area of Figure 1a below is $96 + 14 - 1 = 109$ square units, and the area of Figure 2a is $21 + 26 - 1 = 46$.

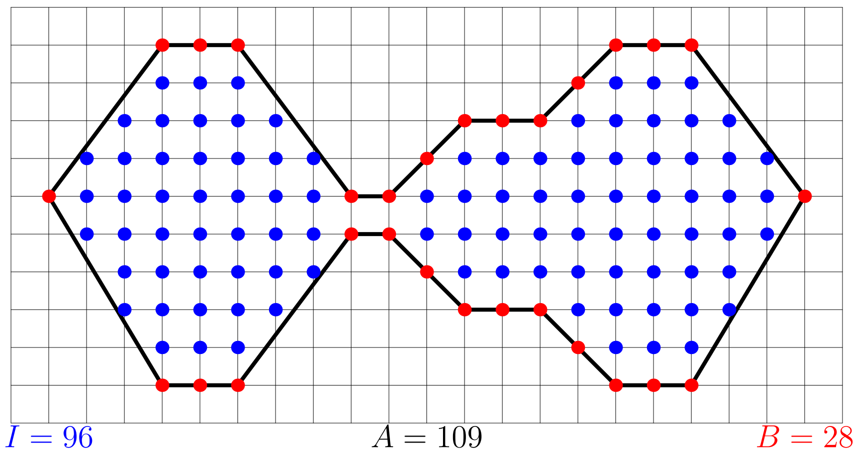


Figure 1a. Figure 1 with calculation.

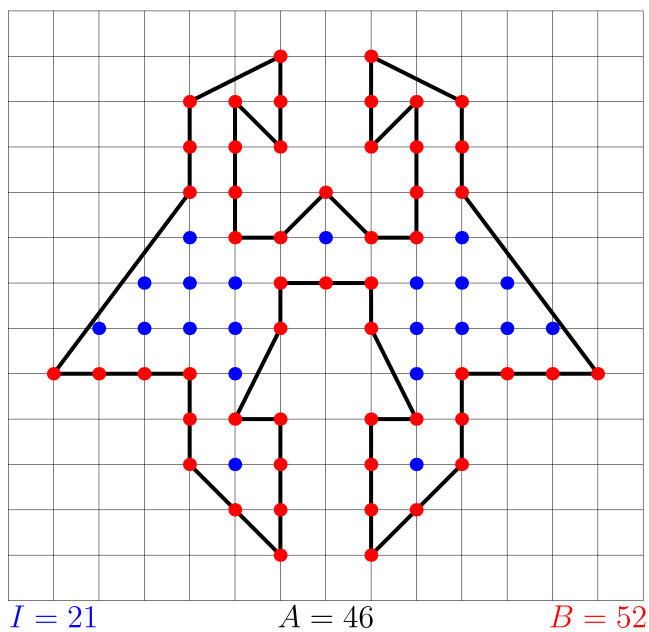
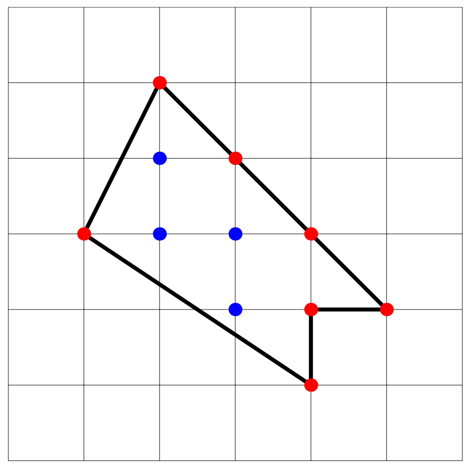


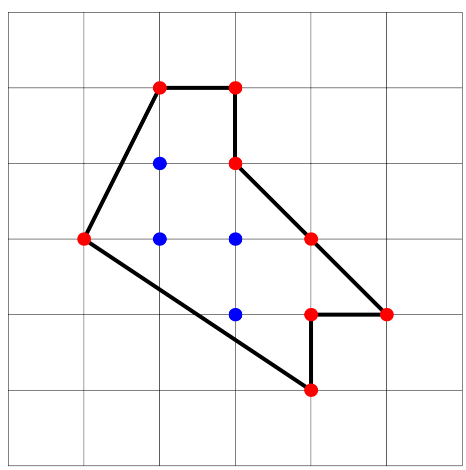
Figure 2a. Figure 2 with calculation.

Incredible, isn't it? Numerous other methods require very specific details about the polygon to calculate its area, but Pick's theorem allows us to calculate it with two simple quantities. (Figure 3, Totoro, will be addressed in Section 4.) This formula also has an interesting implication: if the number of boundary points of a polygon stays constant but an extra interior point is added, then the area definitely increases by exactly 1, and if the number of interior points remains constant but an extra boundary point is added to the polygon, then the area increases by exactly 0.5. This means that Pick's theorem is not only useful for calculating the area of polygons themselves, but also for quickly calculating the area *difference* between two polygons.



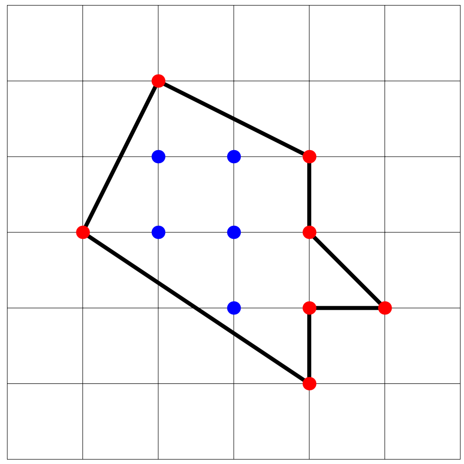
$I = 4$ $A = 6.5$ $B = 7$

Figure 4. A sample polygon with area 6.5.



$I = 4$ $A = 7$ $B = 8$

Figure 5. Increasing the number of boundary points by 1 but keeping the interior intact increases the area by exactly 0.5.



$I = 5$ $A = 7.5$ $B = 7$

Figure 6. Increasing the number of interior points by 1 but keeping the boundary intact increases the area by exactly 1.

Let us now provide proof for Pick's theorem. One proof relies on showing that every such polygon can be dissected into triangles, that Pick's theorem works for triangles, and that Pick's theorem works when generalizing Pick's theorem into higher polygons. Another proof uses graph theory and Euler's theorem, which we explain in Section 3.

3 Proof of Pick's Theorem

Firstly, any polygon P can be dissected into primitive lattice triangles (a primitive lattice triangle being a lattice triangle with no interior points and only 3 boundary points—its vertices); this is intuitively clear, but can be shown formally by showing that any lattice polygon can be dissected into lattice triangles and then showing that all lattice triangles can be dissected into primitive lattice triangles. Next, we can view such a dissection of a simple lattice polygon as a connected planar graph G (that is, the vertices of G are the lattice points in and on P and the edges of G are the sides of the primitive lattice triangles that P is dissected into).

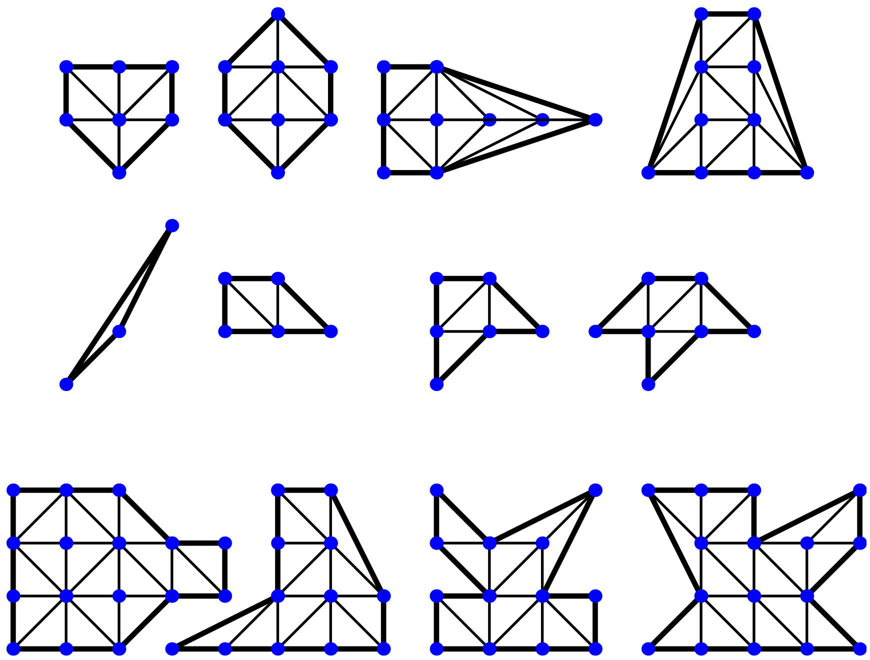


Figure 7. Several graphs illustrating the necessary subdivisions.

It can be shown that the area of any primitive lattice triangle is equal to 0.5. This is trivial by Pick's theorem, but as we will use it in our proof to Pick's theorem, we can prove it by other means; see Area of a Primitive Lattice Triangle in the Appendix for the proof.

Now we have dissected any lattice polygon P into a collection of primitive lattice triangles, which can be interpreted as a planar graph G. Let v be the number of vertices of G, e be the number of edges of G, and f be the number of faces of G. Note that the area of P is equal to the sum of the areas of the primitive lattice triangles; each primitive lattice triangle is a face of our graph G, but there is one other face counted in f - the rest of the plane (the region outside of P). Thus, there are exactly $f-1$ primitive lattice triangles. As each one has area 0.5, the area of P (let this be A) is equal to $\frac{f-1}{2}$.

Euler's formula tells us that we must have $v - e + f = 2$.

We have $f-1$ triangles, and each one has 3 sides; this suggests that we have $3(f-1)$ edges. However, the sides in the interior of G/P are double-counted, so we know that $3(f-1) = e_b + 2e_i$ if e_b is the number of edges on the boundary of G/P and e_i is the number of edges inside G/P (so that $e_b + e_i = e$). We can rewrite what we know as $e = e_b + \frac{3(f-1)-e_b}{2} = 1.5f - 1.5 + 0.5e_b$. We can substitute this value into Euler's formula, and we find that $v - (1.5f - 1.5 + 0.5e_b) + f = v - 0.5f + 1.5 - 0.5e_b = 2 \implies v - 0.5f - 0.5e_b = 0.5 \implies f = 2v - e_b - 1$. Finally, the area of P is equal to $\frac{f-1}{2} = \frac{2v - e_b - 1 - 1}{2} = v - \frac{e_b}{2} - 1$, where v is the number of total vertices of the graph and e_b is the total number of boundary edges of the graph. Let B denote the number of boundary vertices and I denote the number of interior vertices. Note that $e_b = B$ and $v = B + I$; thus, the area of P is equal to $B + I - \frac{B}{2} - 1 = I + \frac{B}{2} - 1$. QED. $\square$

4 Extension of Pick's Theorem

Pick's theorem has a particularly nice extension: it can be extended to also account for holes in simple lattice polygons. As seen in the Totoro example in Figure 3, suppose that an artist would like to determine the area of the gray shaded region in the piece of art. How would the artist do so? Again, countless tedious methods such as triangulation are possible, but we can actually apply a variant of Pick's theorem again.

In particular, we define a *hole* in a simple lattice polygon to be another simple lattice polygon strictly within the lattice polygon that is cut out. Note that the area of such a figure with cut-outs would simply be the total area of the cut-outs subtracted from the area of the entire region; we can express this in the following way:

$$A_{\text{polygon}} = A_{\text{whole}} - \sum A_{\text{hole}}.$$

Further, as the whole region and each hole are all simple lattice polygons, we can apply Pick's theorem to all of them. Number the holes from 1 to h where there are h holes in total, and let I_i and B_i denote the number of interior points and boundary points for hole i respectively for i between 1 and h:

$$\begin{aligned}
A_{\text{polygon}} &= I_{\text{whole}} + \frac{B_{\text{whole}}}{2} - 1 - \sum_{i=1}^{h} \left(I_i + \frac{B_i}{2} - 1 \right) \\
&= I_{\text{whole}} + \frac{B_{\text{whole}}}{2} - \sum_{i=1}^{h} I_i - \frac{1}{2} \sum_{i=1}^{h} B_i - \sum_{i=1}^{h} (-1) - 1 \\
&= I_{\text{whole}} + \frac{B_{\text{whole}}}{2} - \sum_{i=1}^{h} I_i - \frac{1}{2} \sum_{i=1}^{h} B_i + h - 1
\end{aligned}$$

Note that the total number of boundary points of our polygon (with holes) is equal to the sum of the number of boundary points for the whole outline and the sum of the number of boundary points for each of the holes; that is,

$$B_{\text{polygon}} = B_{\text{whole}} + \sum_{i=1}^{h} B_i.$$

The number of interior points is slightly more tricky. It is the total number of interior points for the outline polygon, minus the summed number of boundary points AND interior points for the holes. (This is because, with the addition of holes, the points within the holes are clearly not included, and the points on the holes count as boundary points and not interior points.) In other words:

$$I_{\text{polygon}} = I_{\text{whole}} - \sum_{i=1}^{h} (B_i + I_i).$$

From these two equations, we can obtain expressions for B_{whole} and I_{whole}, which we can then substitute these expressions into our main equation:

$$\begin{aligned}
A_{\text{polygon}} &= I_{\text{whole}} + \frac{B_{\text{whole}}}{2} - \sum_{i=1}^{h} I_i - \frac{1}{2}\sum_{i=1}^{h} B_i + h - 1 \\
&= I_{\text{polygon}} + \sum_{i=1}^{h}(B_i + I_i) + \frac{B_{\text{polygon}} - \sum_{i=1}^{h} B_i}{2} - \sum_{i=1}^{h} I_i - \frac{1}{2}\sum_{i=1}^{h} B_i + h - 1 \\
&= I_{\text{polygon}} + \sum_{i=1}^{h} B_i + \sum_{i=1}^{h} I_i + \frac{B_{\text{polygon}}}{2} - \frac{1}{2}\sum_{i=1}^{h} B_i - \sum_{i=1}^{h} I_i - \frac{1}{2}\sum_{i=1}^{h} B_i + h - 1 \\
&= I_{\text{polygon}} + \frac{B_{\text{polygon}}}{2} + h - 1.
\end{aligned}$$

This modified formula is very similar in form to our original Pick's theorem – it takes the number of interior points and boundary points of some lattice polygon as inputs and gives the area, except now it requires the number of holes. Again, even in this extension, the beauty and simplicity of Pick's theorem pervade: it may seem that including arbitrary holes necessitates specific, complex calculations, but we have just shown that much of it cancels out and disappears. The result only requires 3 simple quantities to calculate the area.

Let us now apply this modified formula with holes to the Totoro polygon.

Using our PickDraw tool, introduced in the next section, we find that the Totoro figure in Figure 8 below has 147 interior points, 186 boundary points, and 7 holes in total. Applying our modified Pick's theorem, we calculate an area of A $= 147 + 58 + 7 - 1 = 211$.

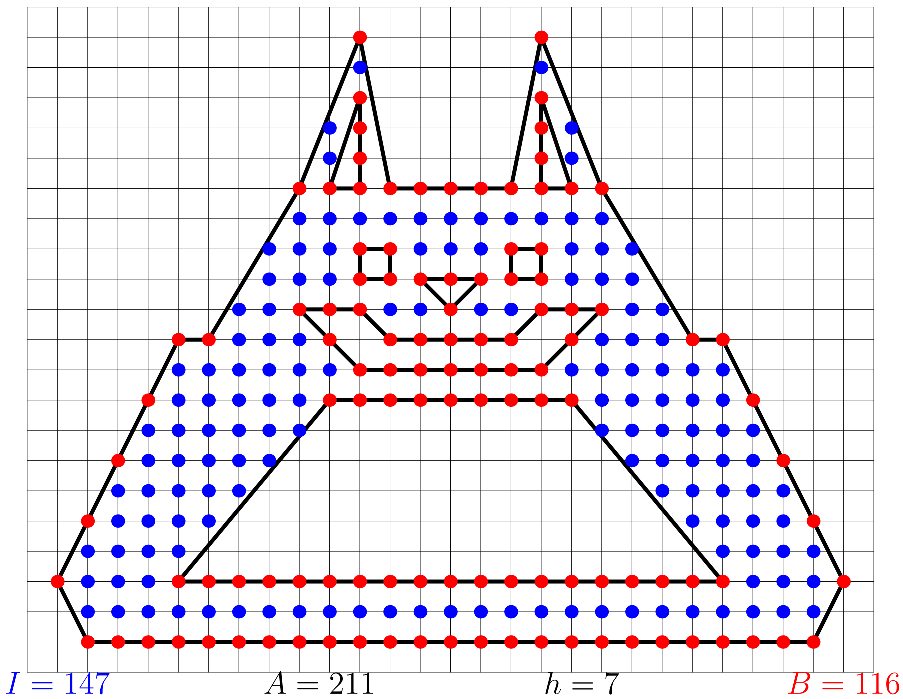


Figure 8. Totoro polygon, labeled with colored interior and boundary points.

Aside from holes, Pick's theorem also has various other extensions. For example, it is valid for not only square lattices such as the Cartesian plane but also other lattices such as triangular (sheared) grids. In a "grid" composed of identical unit equilateral triangles, for instance, we can imagine a polygon whose vertices are all points on the grid (see Figure 9).

The polygon in Figure 9 has 6 boundary points and 3 interior points. Pick's theorem tells us that the area of a polygon with these parameters in rectangular coordinates would be $3 + 0.5 \cdot 6 - 1 = 5$. In this particular lattice grid, the distance between each adjacent lattice point is exactly 1; the x-axis remains unchanged, but the y-axis is simply "squished" (dilated) by a factor of $\frac{\sqrt{3}}{2}$; thus, this polygon's area is $5 \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$.

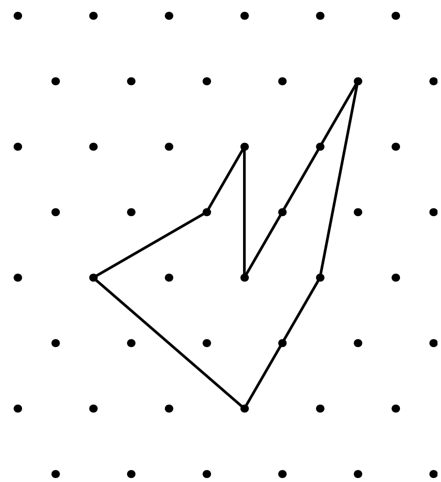


Figure 9. A sheared triangular lattice grid with a polygon.

One might think that requiring vertices to lie on lattice points is too restrictive. However, the lattice point constraint in Pick's theorem can be removed relatively easily. By making the grid arbitrarily small and positioning it correctly, any closed region can be approximated as a set of lattice point vertices and edges.

5 PickDraw: A Tool for Learning and Experimenting With Pick's Theorem

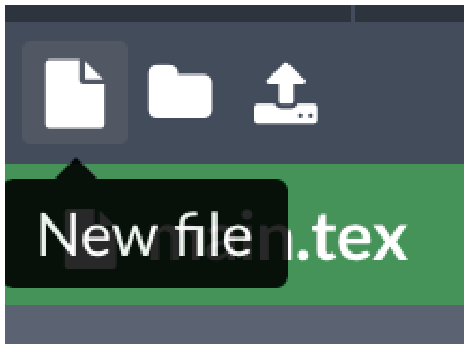
Many aspiring mathematicians will discover the joys of Pick's theorem and lattice point geometry but may find it difficult to draw complex lattice polygons, create digital visuals of their fun images, or calculate the number of points. To facilitate learning and exploring Pick's theorem and lattice point geometry, we have developed PickDraw, which is a simple tool to simplify visualization and calculation for Pick's Theorem: in fact, PickDraw was used to generate the figures in this paper. The source code can be found at

https://github.com/jondoglover/pickDraw/blob/main/pickDraw.asy.

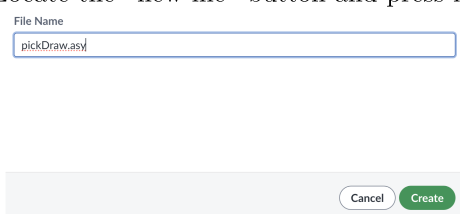
Studies have shown that visuals make learning much more engaging and content much more memorable and understandable. For example, visuals can increase information retention from 10% to 65% three days later (Kane and Pear 2016).

PickDraw uses Asymptote in the LaTeX environment. Since the vast majority of papers are typeset in some form of T_EX or another and the vector graphics language Asymptote goes hand-in-hand with TeX, many students and professors should find PickDraw to be accessible even for novice users. Usage instructions for PickDraw are simple! The setup is to first download the source file "pickDraw.asy" from GitHub and upload it to the local directory of the document (alongside main.tex or the similar document).

The following images provide explanations for Overleaf users.



Locate the "new file" button and press it.



Name the file "pickDraw.asy".

Paste the source code for pickDraw into the file.

In the Asymptote environment in main.tex file, in the same manner as "import graph;", "import pickDraw;" can be added, and PickDraw is ready for use. (Usage assumes an elementary knowledge of Asymptote programming, which is not difficult at all - see this very handy tutorial to get started).

Given the path of a polygon (and optional holes), PickDraw will draw the polygon (and associated holes), label interior and boundary points, display the number of interior/boundary points, and automatically calculate and display the area (the displays of which can be toggled on or off). Usage is simple: for a start, try "poly((1,0)–(5,0)–(7,1)–(6,2)–(7,3)–(7,6)–(6,7)–(5,7)–(5,5)–(2,5)–(2,6)–(1,7)–(0,7)–(0,2)–(1,1)–cycle);". This will yield the following image:

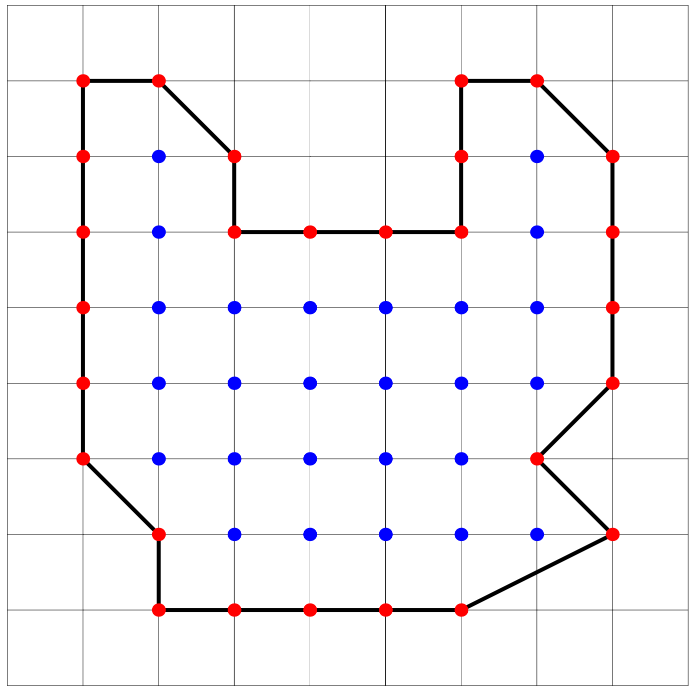


Figure 10. An example of a basic use of PickDraw.

Here's the full code:

```
import graph;
import pickDraw;
unitsize(1cm);

poly((1,0)--(5,0)--(7,1)--(6,2)--(7,3)--(7,6)--(6,7)--(5,7)--(5,5)
--(2,5)--(2,6)--(1,7)--(0,7)--(0,2)--(1,1)--cycle);
```

The *poly* method takes one required argument: the outline of the polygon. In Asymptote, the object is specifically a *path*, and it must be a closed path (hence the –cycle at the end). Another parameter is called *dispIB* and is, by default, set to true, which displays information about the graph (the number of interior points, boundary points, the area, and the number of holes if it is nonzero). The styling of the interior points and the boundary points can also be specified; these are achieved by the inputs *interior* and *boundary*, respectively. Asymptote has a wide range of pre-defined colors; by default, they are `blue+5` and `red+5`, respectively (the 5 indicates width). Finally, there is another optional argument that can be added to account for holes!

Imagine that we wanted to add two holes that can be represented by the triangle with vertices at $(1, 2)$, $(2, 2)$, and $(2, 1)$ and the triangle with vertices at $(3, 3)$, $(4, 3)$, and $(4, 1)$. Then we can add a `holes` parameter like so:

```
import graph;
import pickDraw;
unitsize(1cm);

path[] polygonHoles = {
    (1,4)--(3,4)--(2,3)--cycle,(6,4)--(5,3)--(6,5)--cycle
};
poly(
    (1,0)--(5,0)--(7,1)--(6,2)--(7,3)--(7,6)--(6,7)--(5,7)--(5,5)--(2,5)--(2,6)--(1,7)--(0,
    holes=polygonHoles
);
```

This renders the following:

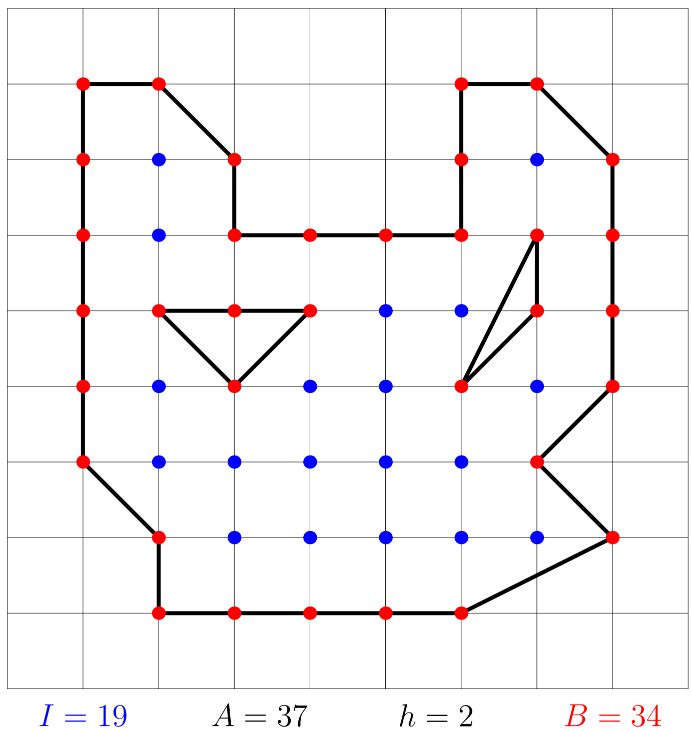


Figure 11. An example of a funny fox drawn with PickDraw.

We can verify the area of this polygon: we can count 34 boundary points (highlighted in red by PickDraw) and 19 interior points (highlighted in blue by PickDraw). There are two holes. Pick's Theorem tells us that the area is $19 + \frac{34}{2} + 2 - 1 = 37$ - which is given by PickDraw and can also be verified with geometry!

6 Interesting Application of Pick's Theorem

Pick's theorem has numerous applications to fields across mathematics. One such application is to Farey sequences. The Farey sequence of order n (written as F_n) is composed of all of the fractions between 0 and 1 that, when fully reduced, have denominator less than or equal to n, sorted in increasing order. (0 is treated as $\frac{0}{1}$ and 1 is treated as $\frac{1}{1}$.) For instance, here are the first few Farey sequences:

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$
$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$F_5 = \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$
$$F_6 = \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{5}, \frac{1}{1}\right\} \ldots$$

And so on. There are several core properties of Farey sequences that build on each other:

- Firstly, it is clear that F_m contains F_n for all $m \geq n$.
- Second, the number of terms in the Farey sequence of order n is exactly $\phi(n)$ more than the number of terms in the Farey sequence of order $n-1$ (that is, from F_{n-1} to F_n, exactly $\phi(n)$ terms are added) where ϕ denotes the elementary Euler totient function.

  (Note that $\phi(n)$ counts the number of positive integers from 1 to $n-1$ that are relatively prime to n; for example, $\phi(5) = 4$ because 1, 2, 3, and 4 are all relatively prime to 5, while $\phi(6) = 2$ because only 1 and 5 are relatively prime to 6.)

  This can be shown by simple reasoning: any fractions in F_n but not F_{n-1} are those with denominator n, and there are exactly $\phi(n)$ fractions with denominator n and a numerator relatively prime to n.
- Third, let the mediant fraction of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ be equal to $\frac{a+b}{c+d}$. For any three consecutive terms in a Farey sequence, the middle term is equal to the mediant fraction of the first and third term.

For any two consecutive terms $\frac{y_1}{x_1}$ and $\frac{y_2}{x_2}$ in F_n, we can calculate the value of $y_2x_1 - y_1x_2$; for $\frac{3}{5}$ and $\frac{2}{3}$ in F_5, then we have $2 \cdot 5 - 3 \cdot 3 = 1$. For $\frac{1}{3}$ and $\frac{1}{2}$ in F_3, we have $1 \cdot 3 - 1 \cdot 2 = 3 - 2 = 1$. Observe that no matter which pairs of fractions we pick, the value always seems to be 1; we can actually prove this using Pick's theorem, even though it seems to be on a different topic!

Firstly, consider the triangle T with vertices at the origin $O = (0,0)$, $A = (x_1, y_1)$, and $B = (x_2, y_2)$. We shall show that the area of T is equal to $\frac{1}{2}(y_2x_1 - y_1x_2)$.

Firstly, it is a well-known geometric fact that the area of a triangle ABO is equal to $\frac{1}{2} \cdot BO \cdot AO \sin(\angle AOB)$. (One can see this by dropping the perpendicular from either A or B and applying $A = \frac{1}{2}bh$.) Here, we have $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $O = (0,0)$.

Let X be some arbitrary point on the positive x-axis. Let $\angle AOX = \alpha$, $\angle BOX = \beta$, and $\angle BOA = \theta$.

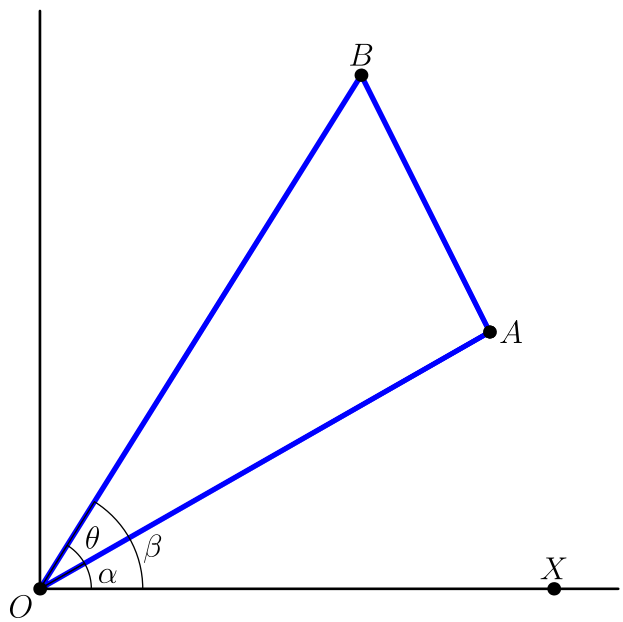


Figure 12. Diagram to illustrate the area proof.

We know that $\theta = \beta - \alpha$, and the area of this triangle is

$$\frac{1}{2}BO \cdot AO \cdot \sin\theta = \frac{1}{2}BO \cdot AO \cdot \sin(\beta - \alpha) = \frac{BO \cdot AO}{2}(\sin\alpha\cos\beta - \sin\beta\cos\alpha).$$

If we drop a perpendicular from A to the x-axis, then we can see that

$\sin\alpha = \dfrac{y_1}{AO}$ and $\cos\alpha = \dfrac{x_1}{AO}$. Similarly, a perpendicular from B to the x-axis yields $\sin\beta = \dfrac{q_2}{BO}$ and $\cos\beta = \dfrac{p_2}{BO}$.

Substituting these values in, we have that the area is

$$\frac{BO \cdot AO}{2}\left(\frac{q_1}{AO}\frac{p_2}{BO} - \frac{q_2}{BO}\frac{p_1}{AO}\right) = \frac{p_2 q_1 - p_1 q_2}{2}.$$

This is the result we wanted.

Now, we shall show that the area of T is also equal to 0.5.

Clearly, there are no points on OA or OB aside from 0, A, and B, since $\gcd(x_1, y_1) = \gcd(x_2, y_2) = 2$. We can also show that there are no points inside T nor on side AB; briefly put, one can use contradiction assuming that some point $P = (a, b)$ exists either on AB or in the interior of T. It follows that the segment OP lies in between OA and OB, so the slope of OP lies in between OA and OB; in other words, $\dfrac{b}{a}$ lies in between $\dfrac{y_1}{x_1}$ and $\dfrac{y_2}{x_2}$. Also, since x_1 and x_2 are both at most n, then a is also at most n. It is fairly obvious now that $\dfrac{b}{a}$ is also in F_n, and not only that, but it is between $\dfrac{y_1}{x_1}$ and $\dfrac{y_2}{x_2}$ - but this is impossible, as we assumed that the two are adjacent terms. Thus, there exists no such point (a, b).

At this point, we have shown that such a triangle T is a primitive lattice triangle. One can show that the area of all primitive lattice triangles is equal to 0.5 through various methods; however, Pick's theorem is arguably the simplest and fastest approach, as it allows a direct computation of $A = I + \dfrac{B}{2} - 1 = 0 + \dfrac{3}{2} - 1 = \dfrac{1}{2}$. Above, we found that the area of T is equal to $\dfrac{1}{2}(y_2 x_1 - y_1 x_2)$; since $\dfrac{1}{2}(y_2 x_1 - y_1 x_2) = \dfrac{1}{2}$, it immediately follows that $y_2 x_1 - y_1 x_2 = 1$. We are done!

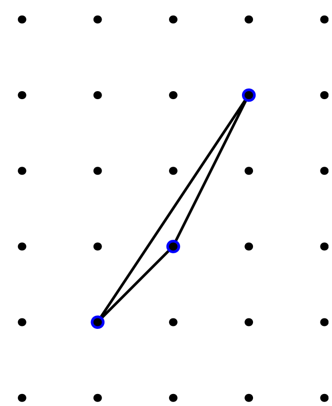
7 Conclusion

Pick's theorem offers a remarkably simple yet powerful formula for calculating the area of lattice polygons, requiring only a count of interior and boundary points. We have offered uniquely visual diagrams and provided a creative proof. As demonstrated, the theorem's elegance lies not only in its straightforward formula but also in its adaptability to more complex cases, such as polygons with holes or non-standard lattices such as sheared grids and and shapes with vertices that are not positioned on lattice points. With a tool like PickDraw, we can make learning and exploring Pick's theorem and lattice point geometry more accessible and engaging. Our exploration also shows how Pick's theorem applications extend to Farey sequences and other mathematical structures. Remarkably, Pick's theorem proves useful in both theoretical mathematics and practical applications.

Appendix

Area of a Primitive Lattice Triangle

From the primitive lattice triangles highlighted in Figure 7, one may hypothesize that all primitive lattice triangles have at least one base that is parallel to the x- or y-axis with length 1, but this is not correct - for example, the following lattice triangle is primitive yet has none of its sides parallel to the x- nor y-axis.



Thus, we require a more nuanced proof than simply stating the height and base to both be 1. An outline is given here. Firstly, we will show that any primitive lattice parallelogram must be spanned by two vectors that also form a basis for $\mathbb{Z}^2$.

Any pair of vectors that forms a basis for $\mathbb{Z}^2$ must be linearly independent, that is, the two vectors are not scalar multiples of each other. Imagine both vectors originating at the origin with heads at points A and B, so the parallelogram spanned by the two vectors has vertices at the origin O, A, and B. Clearly, there must be no lattice points on segments OB and OA other than O, A, and B themselves, since then the vector from O to that supposed point would not be able to be formed by a linear combination of the two vectors. Letting the final vertex of the parallelogram be X, we know that there are similarly no points on AX nor BX aside from A, B, or X because we can imagine translating X to be the origin and applying the same logic. Also, any lattice point P inside the parallelogram would be able to be spanned by a linear combination of the two vectors; we can write $OP = k_1 OA + k_2 OB$ for integers k_1, k_2. However, by definition, $0 < k_1, k_2 < 1$ because P is inside the parallelogram; however, there are no integers strictly between 0 and 1, so there clearly exist no lattice points within the parallelogram. We have shown that the parallelogram contains no interior points nor boundary points aside from its vertices; therefore, the parallelogram must be a primitive lattice parallelogram. This logic also works in reverse.

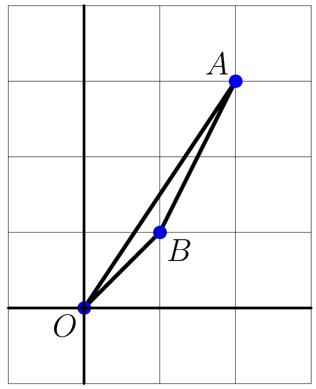


Figure 13. There are no lattice points on nor in this triangle aside from the vertices.

‘ Next, two vectors $\langle a, b\rangle$ and $\langle c, d\rangle$ that form a basis for $\mathbb{Z}^2$ must be able to form any other vector in $\mathbb{Z}^2$ $\langle x_0, y_0\rangle$ via a linear combination of the two; that is, there must exist some integers$m$ and n such that $m\langle a, b\rangle + n\langle c, d\rangle = \langle x_0, y_0\rangle$ for any $x_0, y_0 \in \mathbb{Z}$. One can split this equation into its x- and y-components and manipulate the two equations with elementary algebra to find expressions for m and n in terms of a, b, c, d, x_0, and y_0; these expressions are $\frac{dx_0 - cy_0}{ad - bc}$ and $\frac{ay_0 - bx_0}{ad - bc}$, respectively. As m and n must be integers for all x_0 and y_0, then $ad - bc$ must clearly always be ± 1 to guarantee the integrity of m and n regardless of the values of x_0 and y_0. Then, the area of any primitive lattice parallelogram spanned by two vectors $\langle a, b\rangle$ and $\langle c, d\rangle$ is equal to $ad - bc$ (this can be seen with elementary geometry), which we just showed is 1. Finally, imagine an arbitrary primitive lattice triangle ΔAOB with one vertex O at the origin (we can do this without loss of generality). We can imagine taking a copy of ΔAOB (let it be $\Delta A'O'B'$) and placing vertex A' at B, B' at A, and O' such that O and O' are on opposite sides of the line $AB = B'A'$. This forms parallelogram $OAO'B$ (or $OA'O'B'O$, the two are equivalent).

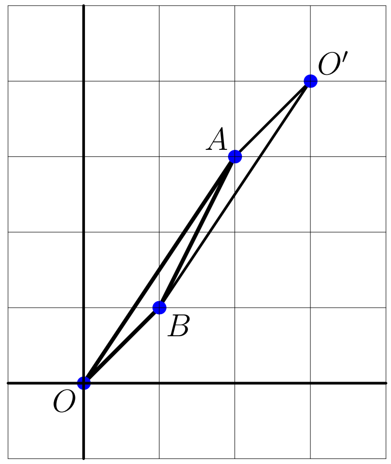


Figure 14. The parallelogram.

Note that as a primitive lattice triangle, ΔAOB contains no lattice points in its interior and only its vertices as boundary points; similarly, $\Delta A'O'B'$ also contains no lattice points and has only its vertices as boundary points. Thus, parallelogram $OAO'B$ is a primitive lattice parallelogram, which has area 1. The area of $OAO'B$ is equal to twice the area of ΔAOB, so the area of $\Delta AOB = \dfrac{1}{2}$.